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Chebyshev Subspaces of L^1 with Linear Metric Projection

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In this note we consider Chebyshev subspaces (i.e., those that contain a unique nearest element to every point) of real $L^1 = L^1[0, 1]$. The result we prove is a characterization of those subspaces which are Chebyshev with linear metric projections (nearest point maps). We also give an example of a Chebyshev subspace whose metric projection is not linear.

There is a paucity of results in our setting. It is known that no subspace of L^1 of finite dimension (see Article IV of [1]) or finite codimension [6] is Chebyshev. In fact, as far as we know, the only Chebyshev subspaces of L^1 known prior to our work were the simple ones constructed as follows. Let $A \subsetneq [0, 1]$ be measurable with positive measure less than one and let $M = \{f \in L^1 : f \text{ vanishes off } A\}$. Then M is Chebyshev with linear metric projection.

On the other hand, there is much known in some related situations. For complex scalars Kahane [3] and others (see [3] for references) have nice results. The situation in which [0, 1] is replaced by a measure space with atoms has also been studied with some success (see [2, 5, 7]).

We mention some terminology. The symbol λ denotes Lebesgue measure on [0, 1]. For f in L^1, denote by $Z(f)$ the set $\{t : f(t) = 0\}$. Then $Z(f)$ is defined only to within a set of measure 0 and set operations involving $Z(f)$ should be interpreted modulo sets of measure 0.

LEMMA. *Let $F \subseteq L^1$ be countable. There exists $g \in V = \overline{sp}\, F$ such that $Z(g) = \bigcap \{Z(f) : f \in F\}$.*

Proof. Let g be a smooth point of the unit ball of V (Mazur's Theorem [4, Satz 2] assures the existence of g). Certainly, $Z(g) \supseteq Z = \bigcap \{Z(f) : f \in F\}$. Suppose $\lambda(Z(g) \backslash Z) > 0$. Choose $f \in F$ such that f does not vanish a.e. on $Z(g) \backslash Z$. Choose $h \in L^\infty$ such that $\| h \| \leqslant 1$, $\int hf\, d\lambda \neq 0$, and h is supported

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on $Z(g)\backslash Z$. Let $\varphi \in V^*$ be the support functional for g and extend φ to a functional of norm 1 on L^1, represented by $h_0 \in L^\infty$. We may assume that h_0 vanishes on $Z(g)$. But now h_0 and $h_0 + h$ are two support functionals for g which are distinct on V (since they differ on f). This contradiction completes the proof.

THEOREM. *Let M be a proper subspace of L^1. Then M is Chebyshev with linear metric projection if and only if M has the following form. There exists a measurable set $A \subseteq [0, 1]$ with $0 < \lambda(A) < 1$ and a linear operator $T: L^1(A) \to L^1(B)$ $(B = [0, 1]\backslash A)$, with $\| Tf \| < \| f \|$ for all nonzero f in $L^1(A)$, such that*

$$M = \{f \in L^1: f|_B = T(f|_A)\}.$$

Proof. Suppose that M has the indicated form. For $g \in L^1$ define Pg to be the element of M which agrees with g on A. Then P is obviously a linear projection onto M. Now let $m \in M$ with $m \neq Pg$. Then

$$\begin{aligned} \| g - Pg \| &= \|(g|_B) - T(g|_A)\| \leqslant \|(g|_B) - (m|_B)\| \\ &\quad + \|(m|_B) - T(g|_A)\| < \|(g|_B) - (m|_B)\| + \|(m|_A) - (g|_A)\| \\ &= \| g - m \|. \end{aligned}$$

Thus M is Chebyshev with metric projection P.

Now suppose M is a Chebyshev subspace with linear metric projection P. Let $M^0 = P^{-1}(0)$. Observe that if $g \in M^0$, $Z(g)$ is a *uniqueness set* for M, i.e., $m \in M$ and $m = 0$ a.e. on $Z(g)$ imply that $m = 0$. To see this, let $h \in M^\perp$ be such that $\| h \| = 1$ and $h(g) = \| g \|$. Then $| h | = 1$ a.e. on $[0, 1]\backslash Z(g)$. Thus every point at which $| h(t)| < \| h \|$ is in $Z(m)$. But the existence of non-zero $h \in M^\perp$ and $m \in M$ satisfying this condition implies that M is not Chebyshev, by Lemma 1 of [7].

Let $r = \inf\{\lambda(Z(f)): f \in M^0\}$. By the lemma, there exists $g_0 \in M^0$ such that $\lambda(Z(g_0)) = r$. Let $A = Z(g_0)$. We see that $0 < \lambda(A) = r < 1$. We claim that

$$M^0 = \{g \in L^1: g = 0 \text{ on } A\}.$$

To prove this, let $g \in M^0$ and suppose g does not vanish on A. By the lemma there exists a linear combination g_1 of g and g_0 such that $Z(g_1) = A \cap Z(g)$ is a proper subset of A. Thus $\lambda(Z(g_1)) < r$, which is impossible. Thus g vanishes on A.

Now suppose $g \in L^1$ and g vanishes on A. Write $g = Pg + g_1$, where $g_1 \in M^0$. Since g and g_1 vanish on A, so does Pg. But A is a uniqueness set for M and therefore $Pg = 0$. Thus $g = g_1 \in M^0$.

Now let $B = [0, 1]\backslash A$ and define $T: L^1(A) \to L^1(B)$ as follows. For f in

$L^1(A)$, extend f to $\bar{f} \in L^1$ by defining $\bar{f}(t) = 0$ for all $t \in B$. Let $Tf = P(\bar{f})|_B$. Then T is linear since P is. Observe also that $P(\bar{f}) - \bar{f} \in M^0$ and so $P(\bar{f})$ agrees with f on A. Thus we have that if $0 \neq f \in L^1(A)$ then $\bar{f} \notin M^0$ and so

$$\| Tf \| = \| \bar{f} - P(\bar{f}) \| < \| \bar{f} \| = \| f \|.$$

Now let $m \in M$ and define $f = m|_A$. Then $\bar{f} - m \in M^0$ and so $m = P\bar{f}$. Thus $T(m|_A) = (P\bar{f})|_B = m|_B$. Conversely, suppose $g \in L^1$ is such that $g|_B = T(g|_A)$. Define $f = g|_A$. Then, as above, $P\bar{f}$ agrees with f and thus g on A. Finally $P\bar{f}$, by definition of T, agrees with $T(\bar{f})$ and thus g on B. Thus $g = P\bar{f} \in M$. This completes the proof.

We remark that if M is a subspace of the form described in the theorem then its metric projection can be described in terms of T as follows. For any $f \in L^1$, Pf is the function which agrees with f on A and $T(f|_A)$ on B.

EXAMPLE. Let

$$M = \{f \in L^1 : f(t + \tfrac{1}{3}) = f(t + \tfrac{2}{3}) = f(t), \qquad \forall\, t \in [0, \tfrac{1}{3})\}.$$

We will show that M is Chebyshev with non-linear metric projection. Observe that the subspace spanned by $(1, 1, 1)$ is Chebyshev in $l^1(3)$. Let $f \in L^1$. For each $t \in [0, \frac{1}{3})$, there is a unique $h(t) \in \mathbb{R}$ which minimizes

$$| f(t) - h(t)| + | f(t + \tfrac{1}{3}) - h(t)| + | f(t + \tfrac{2}{3}) - h(t)|.$$

We will show that h is an integrable function. Once this is done, it is easy to see that the element of M which extends h is the unique best approximation to f in M.

To show that h is measurable, note that h is a composition of measurable functions as follows:

$$h : t \to (f(t), f(t + \tfrac{1}{3}), f(t + \tfrac{2}{3})) \to P[(f(t), f(t + \tfrac{1}{3}), f(t + \tfrac{2}{3}))] \to h(t),$$

where P is the metric projection onto the span of $(1, 1, 1)$ in $l^1(3)$.

To show that h is integrable, note that, for $t \in [0, \frac{1}{3})$.

$$\begin{aligned} | h(t)| &\leqslant | h(t) - f(t)| + | f(t)| \leqslant | h(t) - f(t)| + | h(t) - f(t + \tfrac{1}{3})| \\ &\quad + | h(t) - f(t + \tfrac{2}{3})| + | f(t)| \leqslant 2\, | f(t)| + | f(t + \tfrac{1}{3})| + | f(t + \tfrac{2}{3})|. \end{aligned}$$

The right-hand side is integrable and so h. is.

Finally, we show that the metric projection onto M is not linear. Let f_1 and f_2 be the characteristic functions of $[0, \frac{1}{3})$ and $[\frac{1}{3}, \frac{2}{3})$, respectively.

Both f_1 and f_2 clearly have 0 as best approximation in M. Let h be the constant $\frac{1}{2}$. Then $h \in M$ and

$$\|f_1 + f_2 - h\| = \tfrac{1}{2} < \tfrac{2}{3} = \|f_1 + f_2\|.$$

Hence $f_1 + f_2$ does not have 0 as best approximation.

We have now established that M has the desired properties.

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